A MULTILEVEL SOLVER FOR STOKES EQUATION WITH DISCONTINUOUS VISCOSITY

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Introduction

The goal of this work is to develop an efficient scalable solver for the Stokes problem with a strongly discontinuous viscosity:

$$\begin{cases} (\mu \nabla v, \nabla \phi) + (p, \nabla \circ \phi) & = (g, \phi) \qquad \forall \phi \in \mathbb{V} \\ (q, \nabla \circ v) & = 0 \qquad \forall q \in \mathbb{Q} \end{cases} \tag{1}$$

in domain Ω. The domain Ω is partitioned into subdomains Ω_1 and Ω_2, the coefficient μ is strongly variable and subdomain-wise constant.

The main application of the presented solver are fluid-structure problems [5], where due to large difference of properties between fluid and solid, the apparent viscosity may vary by several orders of magnitude. Other important applications are simulation of viscoplastic fluids flows or composite materials in incompressible elasticy. Also mantle convection problems involve fluid models where viscosity may change by several orders of magnitude [3].

For the standard Stokes problem, there exist efficient linear solvers that also may work on high performance computers. The most popular are Krylov subspace methods combined with block preconditioners [4]. The idea behind block methods comes from dual decomposition of system matrix. It allows one to split the main problem into smaller ones. The sub-problems are easy to solve with well-known techniques such as multigrid or domain decomposition.

However, for the variable viscosity Stokes problem the resulting sub-problems are challenging. Especially, it is non-trivial to obtain the operator spectrally equivalent to the (1,1) block that could be inverted easily. The main problem is the dominating *grad-div* term.

Here we consider multigrid methods for saddle-point problems. The idea is to consider Stokes problem as a symmetric positive-definite problem in divergence-free subspace [1, 2]. Then, the constrained smoother may be constructed and used in the multigrid *V-cycle*. The constrained smoother requires computing projection od divergence-free subspace that is computationally demanding. This can be avioded by using approximate projection [7].

For the Laplace problem with discontinuous coefficient, the Krylov subspace solvers with multilevel preconditioners are robust with respect to the problem size and coefficient jump [6]. One may expect similar performance of GMRes method preconditioned by the constrained multigrid cycle.

The described method is here tested by numerical experiments. The results show uniform covergence with respect to both mesh size and coefficient jump.

Results

Numerical experiments were perfomed to study covergence of presented methods. Geometrical settings is showed on Figure 1. The viscosity on inner square was set to 10^6, on the outer ring viscosity was set to 1. The density $\rho = 1$ on the whole domain. Grids were uniformly refined and used for the test. The iteretive procedure was stopped after reducing residual by 10^{-8}.

Table 1. demostrates the obtained number of iterations. For structured and-semi structured grids, the number of iterations remains constant in all tests. The number of iterations in also comparable the constant coefficient

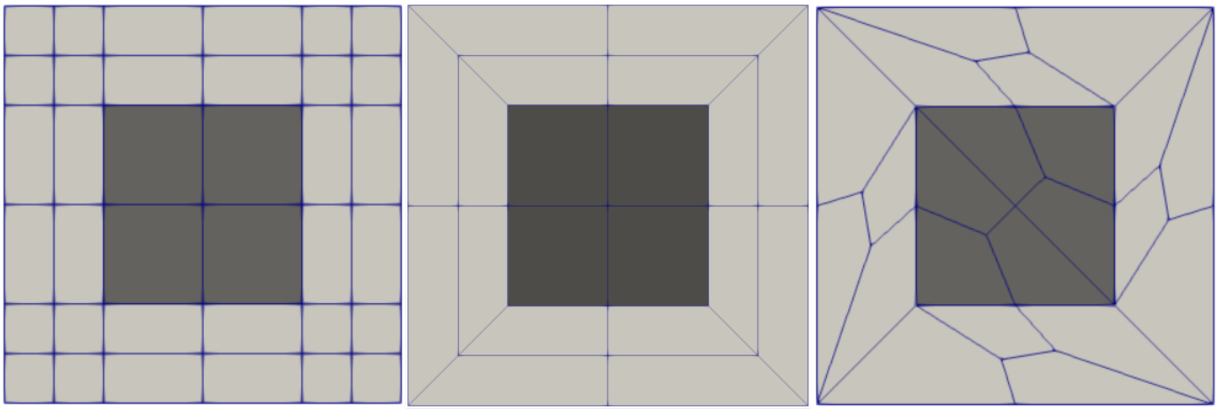

Figure 1: Benchmark problem with coarse grids used for testing the presented algorithm. The inner square is marked in darker grey. From left to right: structured grid, semi-structured grid and unstructured grid.

case. For the ustructured grid, the number of iteration grows slightly, resulting in nearly linear complexity. From the obtained result, one may expect that with increasing problem size the number of iterations may be bounded.

Refinements	Structured grid	Semi-structured grid	Unstructured grid
2	7	11	12
4	7	11	13
8	7	11	15
8, constant coefficient	5	10	13

Table 1: The number of the iterations of preconditioned GMRes method.

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