# ON FUNICULARS AND ARCHGRIDS OF MINIMAL WEIGHT

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## 1. Introduction

The paper discusses the problem of optimum design of roofs and canopies of minimum weight, stressed up to a given limit $\sigma_c$, where $\sigma_c$ is the permissible stress in compression. The structure is subject to a vertically transmissible load and is designed over a plane domain $\Omega$. The structure is designed as a gridwork composed of two families of arches. Its projection on the plane of $\Omega$ forms an orthogonal net which can be identified with the parametric lines $x = \text{const}$ or $y = \text{const}$ of the Cartesian coordinate system. In general, the domain $\Omega$ may be composed of subdomains where the nets are chosen differently. The paper is aimed at an optimal choice of nets to attain the least weight structures. The paper is based on the concept of archgrids proposed by Rozvany and Prager [1].

## 2. The design algorithm

Assume that the coordinates $(x, y)$ of points of the domain $\Omega$ satisfy:

(1) $$x_1(y) \le x \le x_2(y),\ c \le y \le d \quad ; \quad a \le x \le b,\ y_1(x) \le y \le y_2(x)$$

let $l_1(y) = x_2(y) - x_1(y);\ l_2(x) = y_2(x) - y_1(x)$.

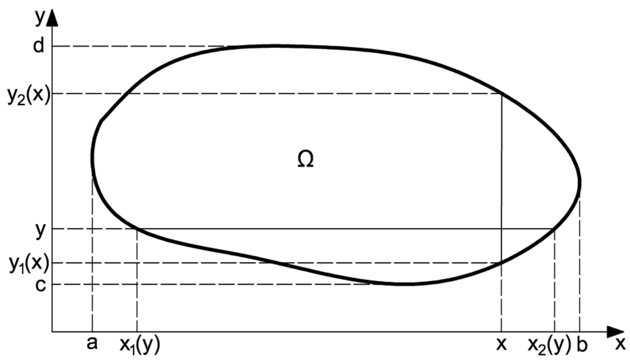


Figure 1: The design domain $\Omega$ over which the archgrid is formed.

The elevation function $z = z(x, y)$ of the roof determines the arch-like slopes of the sections:

- in the section $y = \text{const}$ the arch curve $z = z(x, y)$ links the pin supports at $A_y = (x_1(y), y)$, $B_y = (x_2(y), y)$, where the horizontal reactions are denoted by $H_x(y)$;

-in the section $x = \text{const}$ the arch curve $z = z(x, y)$ links the supports at $C_x = (x, y_1(x))$, $D_x = (x, y_2(x))$ where the horizontal reactions are $H_y(x)$.

The vertical load of intensity $q(x, y)$ is decomposed into two directions: $q_x = \frac{1}{2}q + p$, $q_y = \frac{1}{2}q - p$; $|p| \le \frac{1}{2}q$, where $p = p(x, y)$ is arbitrary at this stage of calculations. We solve the problems of statics of simply supported beams $A_yB_y$, $C_xD_x$ subject to $q_x$, $q_y$, respectively, see [2], [3] . There appear transverse shear forces $\widehat{Q}_x$, $\widehat{Q}_y$ and bending moments $\widehat{M}_x$, $\widehat{M}_y$. The horizontal reactions are expressed by:

(2) $$H_x(y) = \left[ \frac{1}{l_1(y)} \int_{x_1(y)}^{x_2(y)} \left( \widehat{Q}_x \right)^2 \mathrm{dx} \right]^{1/2}$$

and $H_y(x)$ is given in a similar way. These forces become the coefficients of the elliptic equation:

$$H_x(y)\frac{\partial^2 z}{\partial x^2} + H_y(x)\frac{\partial^2 z}{\partial y^2} + q = 0 \tag{3}$$

posed in $\Omega$. The elevation function $z = z(x, y)$ solves (3) with Dirichlet condition $z = 0$ along the contour of $\Omega$. The volume of the structure is given by $V = 2V_1/\sigma_c$ with

$$V_1 = \int_a^b l_2(x)H_y(x)\mathrm{dx} + \int_c^d l_1(y)H_x(y)\mathrm{dy} \tag{4}$$

Since $p(x, y)$ was chosen arbitrary it may not be optimal in a sense that it may not lead to the minimal value of $V$. Let us introduce a new method for which minimum of $V$ will be obtained.

Let us substitute $H_x(y), H_y(x)$ defined by (2) into (4) and denote r.h.s of such equation by $J(\widehat{\mathbf{Q}})$. Note that $J(\widehat{\mathbf{Q}})$ is a positively homogeneous function of degree 1, since $J(\lambda\widehat{\mathbf{Q}})$=$\lambda J(\widehat{\mathbf{Q}})$ if $\lambda \geq 0$. Moreover,

$$J(\mathbf{Q} + \mathbf{P}) \leq J(\mathbf{Q}) + J(\mathbf{P}) \qquad \forall\, \mathbf{P} = (P_x, P_y)\,,\ \mathbf{Q} = (Q_x, Q_y) \tag{5}$$

and, consequently, $J$ satisfies Jensen's inequality and $J$ is convex. The optimization problem reduces to

$$\min\left\{J(\widehat{\mathbf{Q}}) \mid \mathrm{div}\widehat{\mathbf{Q}} + q = 0\right\} \tag{6}$$

If augmented with regularity assumptions, the problem above may be well posed since the functional is convex and the set of admissible $\widehat{\mathbf{Q}}$ is non-empty. Assume that $\widehat{\mathbf{Q}}^* = (\widehat{Q}_x^*, \widehat{Q}_y^*)$ is a minimizer of (6). Then we can compute $H_x, H_y$ by (2) and construct $z$ by solving (3). The function $z$ thus constructed satisfies the mean square slope conditions.

## 3. Final Remarks

In its original formulation of archgrids of minimal weight proposed by Rozvany and Prager [1] the decomposition of the vertical load $q$, at each point of the design domain $\Omega$, into two orthogonal directions is arbitrary. At first the load is decomposed and then for such decomposition the archgrid is constructed. In this paper distribution of the load is not fixed. The minimization over statically admissible shear forces $\widehat{Q}_x, \widehat{Q}_y$ is conducted in order to determine the optimal distribution of the load into two families of perpendicular arches. The numerical algorithm which solves such optimization problem is based on the numerical approach proposed by Czarnecki and Lewiński in [4] in order to solve Free Material Design problem.

**Acknowledgments** The paper has been prepared within the statutory project: Optimum design and computer aided detailing of structures and materials.

**References**

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