THE INTERRELATION OF STATICALLY EXACT AND CONVENTIONAL SHELL THEORIES

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1. Overview

The essential distinctive feature of statically exact and conventional shell theories is the existence of drill moments and drill rotations which are present in the former and absent in the latter. The present contribution immerses into the formulation of prototypical theories, addresses their individual shortcomings, and proposes related remedies in some generality. Therefore it is postulated that both theories shall lead to the same results as long as no external drill moments and no boundary conditions related to the drill rotations are applied. This postulate is supported by the observation that the drill moments can be eliminated from the statically exact balance equations to obtain a set of equations which formally coincide with the ones of the 'canonical' linear shell theory. Remarkably, these equations are obtained *without* the application of any approximate assumption. This considerations lead to a novel constitutive relation which couples the antisymmetric part of the membrane forces and the drill moment components. The parameters involved in this relation can be determined such that the non-standard strains do not produce extra strain energy, which guarantees that the initial postulate is satisfied. It is emphasized that there are no tuning parameters involved. This newly introduced coupling turns out to be essential to achieve correct results for some in-plane bending problems. Further numerical results of prominent benchmark problems confirm the general serviceability of the proposed constitutive law.

2. General

Shell theory has a long history with many doubts and discussions about the 'best' way to obtain an essentially two-dimensional description of an inherently three-dimensional (3D) structure. In particular, two well-established theories have emerged from this process, which are commonly agreed and have proved to be well-suited for the derivation of numerical procedures. In both cases, the shell balance equations are derived from the 3D balance laws giving rise to the definition of resultant shell forces $\mathbf{n}^\alpha$ and moments $\mathbf{m}^\alpha$, where the $\alpha = 1, 2$ refer to the two lateral directions $\mathbf{a}_1$ and $\mathbf{a}_2$. A *statically exact* theory is characterized by the fact that no approximations are involved in this derivation, see [1] and references therein. In contrast, 'conventional' shell theories are of the *first-order shear deformation* (FOSD) type relying on the Reissner-Mindlin kinematic assumption, see [2] for a contemporary treatment of the topic. With other words, warping of transverse material fibres is allowed in the former and prohibited in the latter case, where it gives rise to the definition of a distinguished transverse direction $\mathbf{g}_3$, not necessarily coinciding with the geometric normal direction $\mathbf{a}_3$. In the statically exact case there is no such direction other than $\mathbf{a}_3$ and so-called drill moments $m^{3\alpha} = \mathbf{a}^3 \cdot \mathbf{m}^\alpha$ naturally enter the theory, whilst the relevant transverse component $\mathbf{g}^3 \cdot \mathbf{m}^\alpha = 0$ in the FOSD case. The shell strains $\boldsymbol{\varepsilon}_\alpha$ and curvatures $\boldsymbol{\kappa}_\alpha$ are defined to be the work conjugate counterparts of $\mathbf{n}^\alpha$ and $\mathbf{m}^\alpha$, respectively. They rely on non-symmetric Biot strains in the statically exact and on symmetric Green-Lagrange strains in the FOSD case. The exact role of the kinematic variables is discussed in the presentation. It is frequently ignored that even membrane deformations come along with a warping of transverse material fibers due to the Poisson effect, as long as drill strains $\kappa_{3\alpha} = \mathbf{a}_3 \cdot \boldsymbol{\kappa}_\alpha$ occur. The primary variables are constituted of displacements and rotations. Roughly speaking, three/two rotational degrees of freedom (dofs) are required to specify the orientation of a curved/straight material fiber, respectively. In the latter case, the rotational dofs immediately refer to the orientation of the transverse material fiber indicated by $\mathbf{g}_3$. In the former case, $\mathbf{Q}$ represents an averaged rotation, which does *not* necessarily coincide with the rotation of a specific material point. This is basically the reason why the $\mathbf{g}_3$ is the only feasible choice for the transverse basis vector in FOSD theories. In contrast, for statically exact theories the geometrical normal direction $\mathbf{a}_3$ can serve as a transverse basis vector in a natural way, and the benefits of differential geometry apply. On the other hand, statically exact theories come along with practical drawbacks. In the geometrically linear case, membrane states of stress are commonly considered to be statically determinate. This is no longer the case in the context of theories involving drill moments. Generally, the question arises which constitutive relation applies to the drill moments. Even for the simplest cases this question has remained a matter of ambiguity so far.

3. Equilibrium equations

Referring to the basis $(\mathbf{a}_\alpha, \mathbf{a}_3)$, the balance equations of the statically exact theory read

$$\left\{ \begin{array}{ll} n^{\beta\alpha}|_\alpha - b^\beta_\alpha\, n^{3\alpha} + p^\beta = 0\,, & n^{3\alpha}|_\alpha + b_{\alpha\beta}\, n^{\beta\alpha} + p^3 = 0 \\ m^{\beta\alpha}|_\alpha - n^{3\beta} - \epsilon^{\beta\gamma}\, b_{\alpha\gamma}\, m^{3\alpha} + l^\beta = 0\,, & m^{3\alpha}|_\alpha - \epsilon_{\beta\gamma}\big(n^{\beta\gamma} + b^\beta_\alpha\, m^{\gamma\alpha}\big) + l^3 = 0 \end{array} \right. \tag{1}$$

with p^α, p^3 and l^α, l^3 being the distributed external forces and moments, respectively, $b_{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ the components of curvature and permutation tensor referring to the deformed configuration. The common notations and conventions of Ricci calculus are applied. Introducing *effective* membrane and shear forces, $\hat{n}^{\beta\alpha} := n^{\beta\alpha} - \epsilon^{\mu\alpha}\, m^{3\beta}|_\mu$ and $\hat{n}^{3\alpha} := n^{3\alpha} + \epsilon^{\alpha\mu}\, b_{\beta\mu}\, m^{3\beta}$, respectively, the drill moments $m^{3\alpha}$ can be eliminated and the balance equations transform into

$$\left\{ \begin{array}{ll} \hat{n}^{\beta\alpha}|_\alpha - b^\beta_\alpha\, \hat{n}^{3\alpha} + p^\beta = 0\,, & \hat{n}^{3\alpha}|_\alpha + b_{\beta\alpha}\, \hat{n}^{\beta\alpha} + p^3 = 0 \\ m^{\beta\alpha}|_\alpha - \hat{n}^{3\beta} + l^\beta = 0\,, & \epsilon_{\beta\alpha}\big(\hat{n}^{\beta\alpha} + b^\alpha_\mu\, m^{\beta\mu}\big) + l^3 = 0 \end{array} \right. \tag{2}$$

It is emphasized that the balance equations (2) are fully nonlinear, but formally coincide with the balance equations of the linear FOSD theory, if no external drill moments are applied, i.e. $l^3 = 0$. Note that analogous component equations can *not* be obtained from the geometrically nonlinear FOSD approach, due to the mentioned restrictions concerning the choice of the basis vectors.

4. Constitutive law

For $l^3 = 0$, the effective *pseudo* membrane forces $\hat{\tilde{n}}^{\beta\alpha} := \hat{n}^{\beta\alpha} + b^\alpha_\mu\, m^{\beta\mu}$ are symmetric. Consequently, conventional constitutive equations can be applied for the effective variables. The standard pseudo membrane forces $\tilde{n}^{\beta\alpha} := n^{\beta\alpha} + b^\alpha_\mu\, m^{\beta\mu}$ need not being symmetric any more. For a membrane state of stress and geometric linearity, the symmetric $\hat{\tilde{n}}^{\beta\alpha}$ are uniquely determined by the equilibrium equations (2) and thus coincide with the membrane forces of the conventional theory. This observation together with the definition of the $\hat{\tilde{n}}^{\beta\alpha}$ have an important impact on the possible structure of the constitutive relation from which the antisymmetric part of the membrane forces, $n^{[12]}$, and the $m^{3\alpha}$ result. The simplest possible choice is given by

$$\begin{bmatrix} \tilde{n}^{[12]} \\ m^{31} \\ m^{32} \end{bmatrix} = tG \begin{bmatrix} \beta & \delta\, l_c & -\delta\, l_c \\ \delta\, l_c & \gamma\, l_c^2 & 0 \\ -\delta\, l_c & 0 & \gamma\, l_c^2 \end{bmatrix} \begin{bmatrix} \varepsilon_{[12]} \\ \kappa_{31} \\ \kappa_{32} \end{bmatrix} \tag{3}$$

with G being the in-plane shear modulus. The elastic energy produced by the non-standard strains $\varepsilon_{[12]} := \frac{1}{2}\,(\varepsilon_{12} - \varepsilon_{21})$ and $\kappa_{3\alpha}$ is minimum if the related stiffness matrix is nearly singular, i.e. if $\delta \approx \sqrt{\beta\gamma/2}$. The appearance of a characteristic length l_c reflects the fact that (3) is a localized form of a relation which is, in principle, non-local. The l_c can be chosen as a typical lateral dimension of the problem and is *not* related to the shell thickness t. For finite element calculations, the square root of the element area has turned out to be a feasible choice. Then, the independent, dimensionless parameters can vary in a large range without affecting the numerical results, i.e. $10^{-2} \le \beta \le 10^6$ and $10^{-8} \le \gamma \le 10^{-3}$, as long as δ obeys the relation given above. The parameter β can be interpreted as a penalty parameter enforcing the drill rotation to coincide with the in-plane material rotation, whereas γ plays the role of a (small) stabilization parameter.

References

[1] J. Chróścielewski, J. Makowski, and H. Stumpf. Genuinely resultant shell finite elements accounting for geometric and material non-linearity. *Int. J. Num. Meth. Eng.*, 35:63–94, 1992.

[2] J. C. Simo and D. D. Fox. On a stress resultant geometrically exact shell model. Part I: Formulation and optimal parameterization. *Comput. Methods Appl. Mech. Eng.*, 72:267–304, 1989.