ARE MICHELL STRUCTURES AND OPTIMUM GRILLAGES EQUIVALENT TO FRAMEWORKS COMPOSED OF INFINITE NUMBER OF STRAIGHT MEMBERS?

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A classical engineering problem is that of finding within a plane, convex design domain $\bar{\Omega}$ a truss form that optimally transfers a finite, self-equilibirated system of point-loads F. Since the dawn of the 20th century it has been established that, in general, an optimal truss consisting of finite number of bars does not exist – adding new members proves efficient and the process continues to infinity. The limit framework, called Michell structure, combines two types of media: a continuum-like fibrous domains and cables of finite cross-section area, possibly curved. The aim of this paper is to consider viewing mathematically the Michell structures as trusses made of infinitely many straight bars. The same question will be then readdressed for optimum grillages being a plate-like counterpart of Michell frameworks. We begin by revisiting the well-established description of Michell structures together with the most efficient numerical technique.

The continuum facet of the Michell structures enforces introducing a stress field σ, rather than operating with member forces. The presence of the finitely thick cables, however, rules out the possibility of viewing σ as a function, e.g. belonging to a L^p space; hence σ must be sought in a space of tensor-valued measures $\mathcal{M}\left(\bar{\Omega}; \mathcal{S}^{2\times 2}\right)$. The stress formulation of the Michell problem can readily be posed as:

$$\text{(1)} \qquad \inf\left\{ \int_{\bar{\Omega}} \left(|\sigma_{\mathrm{I}}| + |\sigma_{\mathrm{II}}|\right) : \sigma \in \mathcal{M}\left(\bar{\Omega}; \mathcal{S}^{2\times 2}\right), \operatorname{div}\sigma + F = 0 \right\},$$

where $\sigma_{\mathrm{I,II}}$ are eigenvalues of the stress tensor σ evaluated pointwise. The first, mathematically rigorous treatment of the problem (1) may be found in [2]; in their work authors proved that such variational form attains a solution $\hat{\sigma}$, i.e. the infimum is, in fact, a minimum. The extension of the searched space to measures turns out essential for this result, which validates the physical motivation for capturing the cable part of the structure.

The stress-posed problem (1) admits its dual, displacement-based variational form which was derived by Michell himself in 1904; the functional maximized is a virtual work of the loading system F on a vector-valued displacement function u:

$$\text{(2)} \quad \sup\left\{ \int_{\bar{\Omega}} u \cdot F : u \in C^1\left(\bar{\Omega}; \mathbb{R}^2\right), \max\left\{|\varepsilon_{\mathrm{I}}(x)|, |\varepsilon_{\mathrm{II}}(x)|\right\} \leq 1 \text{ for a.e. } x \in \bar{\Omega}, \varepsilon = 1/2\left(\nabla u + \nabla^{\mathrm{T}} u\right) \right\}.$$

The pointwise constraint bounds all the eigenvalues $\varepsilon_{\mathrm{I,II}}$ of the strain tensor ε. After relaxing the condition $u \in C^1\left(\bar{\Omega}; \mathbb{R}^2\right)$ the problem (2) also has a solution $\hat{u}$.

To this day the most efficient and commonly used method for numerical approximation of Michell structures was first developed in the '60s; it employs the so called *ground structure*. The construction of the ground structure starts by populating the design domain $\bar{\Omega}$ with a finite and fixed set $X \subset \bar{\Omega}$ of evenly spaced m nodes. Next, each pair of nodes in X is connected by a bar, thus generating a dense truss counting $n = m(m-1)/2$ members. The volume minimization problem for the ground structure reduces to a pair of finite-dimensional, mutually dual LP problems:

$$\text{(3)} \qquad \min\left\{ \sum_{i=1}^{n} l_i\,|S_i| : \mathbf{S} \in \mathbb{R}^n,\ \mathbf{B}^{\mathrm{T}}\mathbf{S} = \mathbf{F} \right\},$$

$$\text{(4)} \qquad \max\left\{ \mathbf{u} \cdot \mathbf{F} : \mathbf{u} \in \mathbb{R}^{2m},\ \max_i \frac{|\Delta_i|}{l_i} \leq 1,\ \mathbf{\Delta} = \mathbf{B}\mathbf{u} \right\},$$

where l_i, Δ_i and S_i are, respectively, length, virtual elongation and axial force of the i-th member; $\mathbf{u}$ is a vector of virtual nodal displacements. The geometric matrix $\mathbf{B}$ by definition gives the linear relation between the elongation and displacement vectors $\mathbf{\Delta}$ and $\mathbf{u}$. Together with recently developed adaptive algorithms, for the nodal set X sufficiently "dense", the ground structure approach delivers highly precise numerical representations of the Michell structures with the optimum volume of several significant figures accuracy.

The successful strategy, stemming from the ground structure concept, encourages to pose problems analogical to (3,4) for an infinite nodal set X; we shall consider $X = \bar{\Omega}$. In this setting every pair of points $x \neq y \in \bar{\Omega}$ becomes a potential bar; the ground structure is thus equinumerous to the Cartesian product $\bar{\Omega} \times \bar{\Omega}$, excluding its diagonal. Consequently, instead of operating with finite vectors $\mathbf{\Delta}, \mathbf{S}$, the two-argument functions $\Delta, \lambda : \bar{\Omega} \times \bar{\Omega} \to \mathbb{R}$ must be considered. Unlike perhaps the stress-based problem (3), the kinematic problem (4) can be easily brought (at least formally) to infinite dimensional setting:

$$\sup \left\{ \int_{\bar{\Omega}} u \cdot F : u \in C\left(\bar{\Omega}; \mathbb{R}^2\right), \sup_{(x,y)\in \bar{\Omega}\times\bar{\Omega}} \frac{|\Delta(x,y)|}{|x-y|} \leq 1, \ \Delta = Bu \right\}, \tag{5}$$

where the linear operator B is an infinite dimensional counterpart of the matrix $\mathbf{B}$. This problem was thoroughly examined in [2] and was proven to be equivalent to the relaxed problem (2); hence both problems share the same solution $\hat{u}$. The duality argument yields the stress version of the problem:

$$\inf \left\{ \int_{\bar{\Omega}\times\bar{\Omega}} |x-y| \, |\lambda(x,y)| : \lambda \in \mathcal{M}\left(\bar{\Omega}\times\bar{\Omega}; \mathbb{R}\right), \ B^*\lambda = F \right\}. \tag{6}$$

The member forces λ again, and from the physical point of view precisely for the same reason as σ in (1), must be sought among certain measures instead of functions. Analogies between the forms (3) and (6) are clear; the linear operator B^* is a conjugate of B and, similarly as $\mathbf{B}^{\mathrm{T}}$, should be viewed as an equilibrium operator.

All the aforementioned problems except for (6) are established to have a solution; moreover all the duality gaps are proven to vanish. Therefore, the question of existence of a minimizer $\hat{\lambda}$ of (6) can be interpreted as follows: is there an optimum truss consisting of infinite number of straight bars? A positive answer would render Michell structures as infinite trusses in a strict mathematical sense. The minimizing solution, however, in general fails to exist, which can be intuitively explained: the Michell structures may contain curved bars and no curve is a union (finite or infinite) of straight segments which do not degenerate to points. This settles that Michell structures cannot be identified with infinite trusses.

In the early '70s an intensive research began on the problem of optimum grillages, i.e. horizontal frameworks that transfer the load through beams by means of bending (see e.g. [3]). The continuous setting of the optimization problem reduces to a pair of mutually dual forms analogical (up to second-order differentiation in grillages) to Michell problems (1) and (2). Through numerous, analytically derived optimum grillage layouts a feature that distinguishes them from Michell structures was discerned: for a sufficiently large design domain $\bar{\Omega}$ (in particular when $\bar{\Omega} = \mathbb{R}^2$) the optimum grillage does not contain curved beams. Recently the ground structure approach was successfully employed for the grillage problem, i.e. the analogues of (3) and (4) served as a numerical tool for approximating the exact optimum layouts, see [1]. Finally, the existence of solution of problem (6) posed for grillages remains open; the remarkable absence of curved beams in optimum layouts makes the attempt of proving this result worthwhile – unlike the Michell structures, the optimum grillage would turn out to be true straight-member frameworks.

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