WEAKENED HILL-MANDEL CONDITION FOR COMPUTATIONAL HOMOGENISATION OF RANDOM MEDIA

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1. General definitions

One of the principles of the theory of computational homogenisation is the so called Hill-Mandel condition, which ensures energy preservation during scale transition [2]. This criterion, for the commonly known elasticity problems, is usually written as:

$$\Delta E = \Sigma_{ij}\mathcal{E}_{ij} - \frac{1}{\Omega}\int_{\Omega}\sigma_{ij}\varepsilon_{ij}d\Omega = 0, \tag{1}$$

where σ_{ij}, ε_{ij} are microscopic stress and strain fields, respectively, inside the representative volume element (RVE) of the volume Ω and Σ_{ij}, $\mathcal{E}_{ij}$ are averages of these quantities given by:

$$\Sigma_{ij} = \frac{1}{\Omega}\int_{\Omega}\sigma_{ij}d\Omega \tag{2}$$

$$\mathcal{E}_{ij} = \frac{1}{\Omega}\int_{\Omega}\varepsilon_{ij}d\Omega \tag{3}$$

In this short paper the weakened form of the Hill-Mandel criterion is postulated, i.e.:

$$\Delta E = \Sigma_{ij}\mathcal{E}_{ij} - \frac{1}{\Omega}\int_{\Omega}\sigma_{ij}\varepsilon_{ij}d\Omega \leq \epsilon, \tag{4}$$

where the maximum energetic discrepancy ϵ is defined as:

$$\epsilon = \frac{1}{4}(\Sigma^k_{ij} - \Sigma^s_{ij})\mathcal{E}_{ij}. \tag{5}$$

In the above, the $\Sigma^k_{ij}\mathcal{E}_{ij}$ and $\Sigma^s_{ij}\mathcal{E}_{ij}$ terms stand for upper and lower bounds of the macroscopic strain energy density, obtained when loading the RVE with linear kinematic and minimal kinematic boundary conditions, respectively (see eg. [1, 3, 4]).

2. Derivation of maximum energetic discrepancy

It is well known that the equation (1) is preserved, if the RVE is loaded with the macroscopic strain $\mathcal{E}_{ij}$ by means of linear kinematic, periodic (not considered here) or uniform traction boundary conditions (BCs). Linear kinematic BCs ensure that the perturbation of displacements vanishes at the RVE boundary and the *stiff* answer of the RVE is generated (upper bound). Using these BCs the following set of quantities is obtained as a solution: ε^k_{ij}, σ^k_{ij} and Σ^k_{ij}. Uniform traction BCs ensure in turn, that the perturbation of tractions is vanishing at boundary and the *soft* answer of the RVE is obtained (lower bound). These BCs provide the following solution set: ε^s_{ij}, σ^s_{ij} and Σ^s_{ij}. Let's assume now, that there exist a smooth, linear transition from one state of the material to the other and the following intermediate strain field can be defined:

$$\varepsilon^\alpha_{ij} = (1-\alpha)\varepsilon^k_{ij} + \alpha\varepsilon^s_{ij}, \qquad \alpha \in \langle 0, 1\rangle \tag{6}$$

The above can be followed by the definitions of intermediate microscopic and macroscopic stresses:

$$\sigma_{ij}^{\alpha} = (1-\alpha)\sigma_{ij}^{k} + \alpha\sigma_{ij}^{s} \tag{7}$$

$$\Sigma_{ij}^{\alpha} = (1-\alpha)\Sigma_{ij}^{k} + \alpha\Sigma_{ij}^{s} \tag{8}$$

The assumption of smooth transition can be fully justified by the assumption of the continuity of the displacement field and the well behaving material properties (constants for example). Now, let's consider the discrepancy between the macroscopic and microscopic strain energy density for this intermediate state:

$$\Delta E^{\alpha} = \Sigma_{ij}^{\alpha}\mathcal{E}_{ij} - \frac{1}{\Omega}\int_{\Omega}\sigma_{ij}^{\alpha}\varepsilon_{ij}^{\alpha}d\Omega. \tag{9}$$

Inserting equations (6)-(8) into (9) the following is derived, after some transformations:

$$\Delta E^{\alpha} = \alpha(1-\alpha)(\Sigma_{ij}^{k} - \Sigma_{ij}^{s})\mathcal{E}_{ij}, \tag{10}$$

It is immediate to see that the above attains its maximum at $\alpha = 0.5$ and takes the value given by equation (5).

3. Boundary conditions for weakened Hill-Mandel principle

Note that for kinematic solution it can be written at the boundary: $\varepsilon_{ij}^{k} = \mathcal{E}_{ij}$, whereas for the static solution the following holds: $\varepsilon_{ij}^{s} = \mathcal{E}_{ij} + \tilde{\varepsilon}_{ij}^{s}$. The $\tilde{\varepsilon}_{ij}^{s}$ is the strain perturbation at static loading, which vanishes at boundary in the integral sense only. The strain perturbation at kinematic loading $\tilde{\varepsilon}_{ij}^{k}$ also exists, but it just vanishes strictly at boundary. From equation (6) it is then derived, that the intermediate state can be obtained by applying the at the RVE boundary:

$$\varepsilon_{ij}^{\alpha} = \mathcal{E}_{ij} + \alpha\tilde{\varepsilon}_{ij}^{s}. \tag{11}$$

which stand for the *statically perturbed boundary conditions* [4]. Note that the static solution ε_{ij}^{s} have to be known first, in order to apply these boundary conditions.

4. Conclusions and prospects

The weakened Hill-Mandel condition for homogenisation of disordered media is proposed which allows for discrepancy in macro- and microscopic energy density. The maximum discrepancy is derived from the assumption that the true answer of the macroscopic material is located in between the kinematic and static solutions taken from the representative volume element. Such definition opens new possibilities in computational homogenisation, especially it potentially allows for smaller volumes of the material which can serve as the RVE (see results reported in [4]).

References

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